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NOTE

The Particle Strength Exchange Method Applied to Axisymmetric Viscous Flows

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1. INTRODUCTION

The numerical simulation of axisymmetric vortex flows by means of particle methods has been investigated in a previous paper [7]. Although axisymmetric flows are almost impossible to reproduce in industrial or even experimental devices, they have been used extensively as the basic model of many problems of paramount practical interest, such as the vortex breakdown or the modelization of flows in turbomachinery.

In [7], two different methods were proposed for the simulation of diffusion. The first one was based on the diffusion velocity concept, whereas the second was the particle strength exchange (PSE) method extended to axisymmetric flows. In the absence of any mathematical analysis, the design of the smoothing function was achieved by using the Green function of the diffusion equation. There are many different ways to build a PSE method for axisymmetric flows. One way should be to use a cylindrical mapping and a Cartesian form for the smoothing function. However, the diffusion operator has changed in the mapping, and boundary conditions have to be satisfied on the axis. Although it leads to the introduction of a singular convection velocity, one possible solution to this problem is the splitting method proposed by Martins and Ghoniem [4]. This difficulty has been overcome by Martins and Ghoniem by means of an explicit integration of the corresponding term.

In this paper, a full PSE method has been obtained by deriving an integral form for the complete radial diffusion operator. To achieve this goal, we start from a three-dimensional formulation in Cartesian coordinates. The result is then expressed in cylindrical coordinates and integrated in the azimuthal direction. Both cases of scalar and vectorial diffusion have

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been investigated using the general formulation introduced by Mas-Gallic [5]. The purpose of this note is also to investigate further the derivation of the PSE method for these two cases (Sections 3 and 4). There are basically two differences between scalar and vectorial axisymmetric diffusion. First, the diffusion operators expressed in polar coordinates are different, although they have the same expression in a two- or a three-dimensional Cartesian coordinate system. Second, the boundary conditions are also different: The continuity of a scalar field is ensured provided its normal derivative along the axis is zero, whereas a nonaxial vector field has to be zero itself. It has been found that different methods can be obtained depending on the insertion of the discretisation step which can be performed before or after the integration step. More details will be provided in Section 5 below.

2. INTEGRAL SOLUTION OF THE DIFFUSION EQUATION

We start from the smooth particle approximation ϕ_ε of a function which can be either a vector field with only one nonzero azimuthal component $\phi = \phi_\theta$ such as vorticity $\boldsymbol{\omega}$ or a scalar function ϕ:

$$\phi_\varepsilon(x) = \int_V \phi(x')\mathcal{F}_\varepsilon(|\mathbf{x}' - \mathbf{x}|)\, dv(\mathbf{x}'). \tag{1}$$

In this equation, ε is the cutoff number, and $\mathcal{F}_\varepsilon$ is a 3-D radially symmetric regular function of unit weight, whose limit as $\varepsilon \to 0$ is the Dirac measure [2]. The diffusion operator is applied to the funtion ϕ. We denote λ the diffusion coefficient, and we start with the representation of the diffusion operator in a three-dimensionnal space using an integral approximation [5]:

$$\begin{aligned}\boldsymbol{\nabla} \cdot (\lambda(\mathbf{x})\boldsymbol{\nabla}\phi_\varepsilon) \approx \int_V &(\lambda(\mathbf{x}) + \lambda(\mathbf{x}'))(\phi(\mathbf{x}') - \phi(\mathbf{x}))\\ &\times \frac{\boldsymbol{\nabla}\mathcal{F}_\varepsilon(|\mathbf{x}' - \mathbf{x}|) \cdot (\mathbf{x} - \mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|^2}\, dv(\mathbf{x}'). \end{aligned}\tag{2}$$

A second-order three-dimensional Gaussian smoothing function has been used hereafter. Thank to this choice, we were able to derive a close form for the integral (2). We did not investigate other cases, although it is probably possible to obtain similar results by means of algebraic or any other kind of smoothing functions. One of the main advantages of the Gaussian in that case is that it is the exact Green function for the three-dimensional diffusion problem. This property will be used later on. We set

$$\mathcal{F}_\varepsilon(x) = \frac{1}{(\pi\varepsilon^2)^{3/2}} \exp\left(-\frac{x^2}{\varepsilon^2}\right). \tag{3}$$

The gradient of $\mathcal{F}$ can be explicitely computed as

$$\frac{\boldsymbol{\nabla}\mathcal{F}_\varepsilon(x) \cdot\ \mathbf{x}}{x^2} = -\frac{2}{\varepsilon^2}\mathcal{F}_\varepsilon(x). \tag{4}$$

Splitting (2) in two terms yields two integrals which have to be computed, namely,

$$\begin{aligned}\mathbf{h}_1(\mathbf{x}) &= \int_V \phi(\mathbf{x}')\mathcal{F}_\varepsilon(|\mathbf{x}' - \mathbf{x}|)\, dv(\mathbf{x}')\\ \mathbf{h}_2(\ \mathbf{x}) &= \int_V \phi(\mathbf{x})\mathcal{F}_\varepsilon(|\mathbf{x}' - \mathbf{x}|)\, dv(\mathbf{x}') = \phi(\ \mathbf{x}) \int_V \mathcal{F}_\varepsilon(\mathbf{x}' -\ \mathbf{x})\, dv(\mathbf{x}'). \end{aligned}\tag{5}$$

These integrals have been computed for two different situations: ϕ is a scalar, and ϕ is a vector. In this last case it will be shown in Section 5 that two different alternatives exist, leading to two equivalent discrete formulations. For convenience, the computation of the different integrals will be carried out in a cylindrical coordinates system (r, z, θ).

3. PARTICLE APPROXIMATION OF THE DIFFUSION OF AN AXISYMMETRIC SCALAR FIELD

In the case where ϕ is a scalar field, it does not depend on the azimuthal coordinate θ. An approximation ϕ_ϵ for ϕ can be readily obtained by using the previously defined Gaussian smoothing function $\mathcal{F}_\epsilon$ and the particle weight $\Phi \simeq \phi\delta S$, which is a discrete form for the differential $\phi\, dr dz$:

$$\begin{aligned}\phi_\varepsilon(r, z) &= \int_V \phi(r', z')\mathcal{F}_\epsilon(|x - x'|)r'\, dr'\, dz'\, d\theta \\ &= \int_{\mathcal{S}} \phi(r', z')\left(\int_\theta \mathcal{F}_\epsilon(|x - x'|)r'\, d\theta\right) dr'\, dz' \\ &= \int_{\mathcal{S}} \phi(r', z')\mathcal{G}_{0\epsilon}(r, z, r', z')\, dr'\, dz'. \end{aligned} \tag{6}$$

This is the result of the integration in the plane (r, z) of Eq. (1). The term $\mathcal{S}$ is the semi-infinite meridian plane and

$$\mathcal{G}_{0\varepsilon}(r, z, r', z') = \frac{2r'}{\sqrt{\pi}\varepsilon^3} \exp\left\{ -(r'^2 + r^2 + (z' - z)^2)/\varepsilon^2\right\} I_0\left(\frac{2r'r}{\varepsilon^2}\right), \tag{7}$$

where I_i is the modified Bessel function of ith order and with

$$\int_{\mathcal{S}} r'\mathcal{G}_{0\varepsilon}(r, z, r', z')\, dr'\, dz' = r. \tag{8}$$

Relation (7) ensures zero flux across the axis ($\frac{\partial\phi}{\partial r}|_{r=0} = 0$), and the integral result for the function $\mathcal{G}_{0\varepsilon}$ (Eq. (8)) ensures that the total mass is conserved. Discretising the surface $\mathcal{S}$ using particles with $\Phi \approx \phi\delta S$, we obtain the discrete form:

$$\phi_\varepsilon(r, z) = \sum_i \Phi_i \mathcal{G}_{0\varepsilon}(r, z, r_i, z_i). \tag{9}$$

The diffusion operator can be computed as well by using a similar procedure. The first integral $\mathbf{h}_1(\mathbf{x})$ was evaluated in [7]. The second one can be computed accordingly,

$$\nabla \cdot (\lambda(r, z)\nabla\phi_\varepsilon(r, z)) = \frac{4}{\varepsilon^2}\int_{\mathcal{S}} r'\overline{\lambda}\{\phi(r', z') - \phi_\varepsilon(r, z)\}\mathcal{H}_{0\varepsilon}(r, z, r', z')\, dr'\, dz', \tag{10}$$

with $\overline{\lambda} = (\lambda(r, z) + \lambda(r', z'))/2$ and we note $\mathcal{H}_{i\varepsilon}$ the function

$$\mathcal{H}_{i\varepsilon}(r, z, r', z') = \frac{2}{\sqrt{\pi}\varepsilon^3} \exp[-(r^2 + r'^2 + (z - z')^2)/\varepsilon^2] I_i(2rr'/\varepsilon^2). \tag{11}$$

Discretising $(\mathcal{S})$ with particles Φ_i with elementary surface (S_i) and assuming that ϕ can be approximated by a constant on the particles yield the following equation:

$$\nabla \cdot (\lambda \nabla \phi_\varepsilon) \mid_{(r_i,z_i)} S_i = \frac{4}{\varepsilon^2} \sum_j r_j \frac{(\lambda_i + \lambda_j)}{2} \{\Phi_j S_i - \Phi_i S_j\} \mathcal{H}_{0\varepsilon}(r_i, z_i, r_j, z_j). \quad (12)$$

4. PARTICLE APPROXIMATION OF THE DIFFUSION OF AN AXISYMMETRIC VECTOR FIELD

Similarly, in the case where ϕ is the azimuthal component of the vector field: $\phi = \phi_\theta$, we get

$$\begin{aligned} \phi_\varepsilon(r, z) &= \int_V \phi(r', z') \cos(\theta) \mathcal{F}_\varepsilon(|x - x'|) r' \, dr' \, dz' \, d\theta \\ &= \int_{\mathcal{S}} \phi(r', z') \left(\int_\theta \cos(\theta) \mathcal{F}_\varepsilon(|x - x'|) r' \, d\theta \right) dr' \, dz' \\ &= \int_{\mathcal{S}} \phi(r', z') \mathcal{G}_{1\epsilon}(r, z, r', z') \, dr' \, dz'. \end{aligned} \quad (13)$$

The function $\mathcal{G}_{1\varepsilon}$ has been derived in order to satisfy the boundary conditions, that is, zero on the axis. Thus we get the expression

$$\mathcal{G}_{1\varepsilon}(r, z, r', z') = \frac{2r'}{\sqrt{\pi} \varepsilon^3} \exp\{-(r'^2 + r^2 + (z' - z)^2)/\varepsilon^2\} I_1\left(\frac{2rr'}{\varepsilon^2}\right). \quad (14)$$

Irrespective of the value of ε, we can verify that $\int_{\mathcal{S}} r'^2 \mathcal{G}_{1\varepsilon}(r, z, r', z') \, dr' \, dz' = r^2$ which ensures the preservation of the first momentum. The discrete form of Eq. (13) is

$$\phi_\varepsilon(r, z) = \sum_i \Phi_i \mathcal{G}_{1\varepsilon}(r, z, r_i, z_i). \quad (15)$$

We turn now to the discretisation of the diffusion operator according to Eq. (2),

$$h_2(r, z) = \int_{\mathcal{S}} \frac{\phi(r, z)}{(\pi \varepsilon^2)^{3/2}} \exp[-(r^2 + r'^2 + (z - z')^2)/\varepsilon^2] I_0(2rr'/\varepsilon^2) 2\pi r' \, dr' \, dz'. \quad (16)$$

Finally, the diffusion operator applied to an axisymmetric vector field is

$$\begin{aligned} \nabla \cdot (\lambda(r, z) \nabla \phi_\varepsilon(r, z)) = \frac{4}{\varepsilon^2} \int_{\mathcal{S}} r' \bar{\lambda} \{ & \phi(r', z') \mathcal{H}_{1\varepsilon}(r, z, r', z') \\ & - \phi(r, z) \mathcal{H}_{0\varepsilon}(r, z, r', z') \} \, dr' \, dz'. \end{aligned} \quad (17)$$

A particle discretisation for this expression can be obtained by using the previously defined Φ_j particle weight,

$$\nabla \cdot (\lambda(r, z) \nabla \phi_\varepsilon(r, z)) = \frac{4}{\varepsilon^2} \sum_j r_j \bar{\lambda} \left\{ \Phi_j \mathcal{H}_{1\varepsilon}(r, z, r_j, z_j) - \Phi \frac{S_j}{S} \mathcal{H}_{0\varepsilon}(r, z, r_j, z_j) \right\}. \quad (18)$$

The discrete form of the diffusion operator applied to particles is

$$\begin{aligned}\mathcal{D}_\phi &= \nabla\cdot(\lambda\nabla\phi_\varepsilon)\mid_{(r_i,z_i)}\mathcal{S}_i\\ &= \frac{4}{\varepsilon^2}\sum_j r_j\frac{(\lambda_i+\lambda_j)}{2}\{\Phi_j S_i\mathcal{H}_{1\varepsilon}(r_i,z_i,r_j,z_j)-\Phi_i S_j\mathcal{H}_{0\varepsilon}(r_i,z_i,r_j,z_j)\}.\end{aligned} \tag{19}$$

If ϕ is the azimuthal component of the vorticity this leads to the following particle strength exchange (PSE) scheme for ϕ:

$$\frac{d\Phi_i}{dt}=\mathcal{D}_\phi. \tag{20}$$

In the midplane the total particle weight is not conserved, $\frac{d}{dt}(\sum_i\Phi_i)<0$. This leak is balanced with the flux of ϕ throught the line $z=0$. For a vorticity field, this is in accordance with the generalised Kelvin theorem [1]. This method can be easily incorporated in a complete Navier–Stokes algorithm using free particles. The velocity induced by a set of vortex rings is evaluated by the desingularised vortex method proposed by Nitsche [6].

5. APPLICATION TO VORTEX FLOWS

In this section, the case of an incompressible unbounded vortex flow is considered. The problem under consideration is the form of the PSE model and the necessity to account explicitely for the vorticity diffusion flux across the axis. A straightforward application of the algorithms of Section 4 enlightens this problem and brings some explanations on the vorticity diffusion for axisymmetric flows. The vorticity field $\omega=\phi_\theta$ satisfies the following diffusion problem if we consider the case of uniform viscosity:

$$\begin{aligned}&\frac{\partial\omega}{\partial t}=\nu\left(\frac{\partial^2\omega}{\partial r^2}+\frac{\partial^2\omega}{\partial z^2}-\frac{\omega}{r^2}+\frac{1}{r}\frac{\partial\omega}{\partial r}\right) && \text{in } [0,+\infty[\times]-\infty,+\infty[\\ &\omega=0 && \text{for } r=0.\end{aligned}$$

A numerical solution of which can be expressed, either using the previous expression

$$\frac{d\Gamma_i}{dt}=\frac{4\nu}{\varepsilon^2}\sum_j r_j\{\Gamma_j S_i\mathcal{H}_{1\varepsilon}(r_i,z_i,r_j,z_j)-\Gamma_i S_j\mathcal{H}_{0\varepsilon}(r_i,z_i,r_j,z_j)\}, \tag{21}$$

or according to the analysis of [7]

$$\Gamma_i(t)=\Gamma_i(0)\left(1-\exp\left(-r_i^2/(4\nu t)\right)\right)+\sum_j\{r_j\Gamma_j S_i-r_i\Gamma_i S_j\}\mathcal{H}_{1\sqrt{4\nu t}}(r_i,z_i,r_j,z_j), \tag{22}$$

where $\Gamma\approx\int_S\omega\,dr\,dz$ is the particle strength. The connection between these two expressions can be derived from mathematical arguments although they correspond to two different ways of derivation on a physical background. In order to establish the last formulation, we have to consider the case of a uniform viscosity. The diffusion operator (Eq. (17)) is

$$\nu\Delta\omega_\varepsilon(r,z)=\frac{4\nu}{\varepsilon^2}\int_S r'\{\omega(r',z')\mathcal{H}_{1\varepsilon}(r,z,r',z')-\omega(r,z)\mathcal{H}_{0\varepsilon}(r,z,r',z')\}\,dr'\,dz'. \tag{23}$$

This equation can be expressed with the notations of Sections 2 and 4 using the functions h_1 and h_2. We then have

$$\nu \Delta\omega_\varepsilon(r, z) = \frac{4\nu}{\varepsilon^2}(h_1(r, z) - h_2(r, z)). \tag{24}$$

We are interested in the function h_2 of this equation:

$$\begin{aligned} h_2(r, z) = \omega(r, z) &\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi\varepsilon^2}} \exp[-(z - z')^2/\varepsilon^2]\, dz' \\ &\times \int_0^{+\infty} \frac{2r'}{\varepsilon^2} \exp[-(r^2 + r'^2)/\varepsilon^2] I_0(2rr'/\varepsilon^2)\, dr'. \end{aligned} \tag{25}$$

We apply to the second integral, which is only dependent in r, an integration by parts,

$$\begin{aligned} &\int_0^{+\infty} \frac{2r'}{\varepsilon^2} \exp[-(r^2 + r'^2)/\varepsilon^2] I_0(2rr'/\varepsilon^2)\, dr' \\ &\quad = \exp(-r^2/\varepsilon^2) + \int_0^{+\infty} \frac{2r}{\varepsilon^2} \exp[-(r^2 + r'^2)/\varepsilon^2] I_1(2rr'/\varepsilon^2)\, dr'. \end{aligned} \tag{26}$$

Substituting this result in Eq. (23) we obtain another expression for the diffusion operator,

$$\begin{aligned} \nu\Delta\omega_\varepsilon(r, z) = &-\frac{4\nu\omega(r, z)}{\varepsilon^2} \exp(-r^2/\varepsilon^2) \\ &+ \frac{4\nu}{\varepsilon^2} \int_{\mathcal{S}} \{r'\omega(r', z') - r\omega(r, z)\} \mathcal{H}_{1\varepsilon}(r, z, r', z')\, dr'\, dz'. \end{aligned} \tag{27}$$

The discrete form of the previous equation using the particle approximation is

$$\mathcal{D}_\omega = -\frac{4\nu\Gamma_i}{\varepsilon^2} \exp\left(-r_i^2/\varepsilon^2\right) + \frac{4\nu}{\varepsilon^2} \sum_j \{r_j\Gamma_j S_i - r_i\Gamma_i S_j\} \mathcal{H}_{1\varepsilon}(r_i, z_i, r_j, z_j). \tag{28}$$

We integrate the transport equation (20) for the circulation attached to a particle using a first-order scheme in time. By replacing the core radius ε by the core radius coming from the solution of heat transfer theory ($\varepsilon = \sqrt{4\nu t}$) we obtain Eq. (22). This result can be obtained in a different way by considering the different term of the diffusion process. A physical interpretation of the basic form of the PSE method for a Gaussian regularisation function has been provided in [3]. The application to axisymmetric flows is the result of two steps. The first one consists of an integration in θ of the three-dimensional diffusion equations, and the second is the reduction of the problem to a two-dimensional one in the meridian plane. The two previous expressions correspond exactly to different orders in performing these two steps.

From a numerical point of view, these two methods are not exactly equivalent because the integration domain is theoretically infinite whereas the numerical domain is not. Therefore, the evaluation of integral (25) is nothing more that an approximation of the sum of the integral and the exponential term of Eq. (26). To evaluate the accuracy of the two methods, the one-dimensional radial problem has been solved for different particle numbers. The results are presented on Figs. 1 and 2. We plotted on Fig. 1 the local error on the vorticity

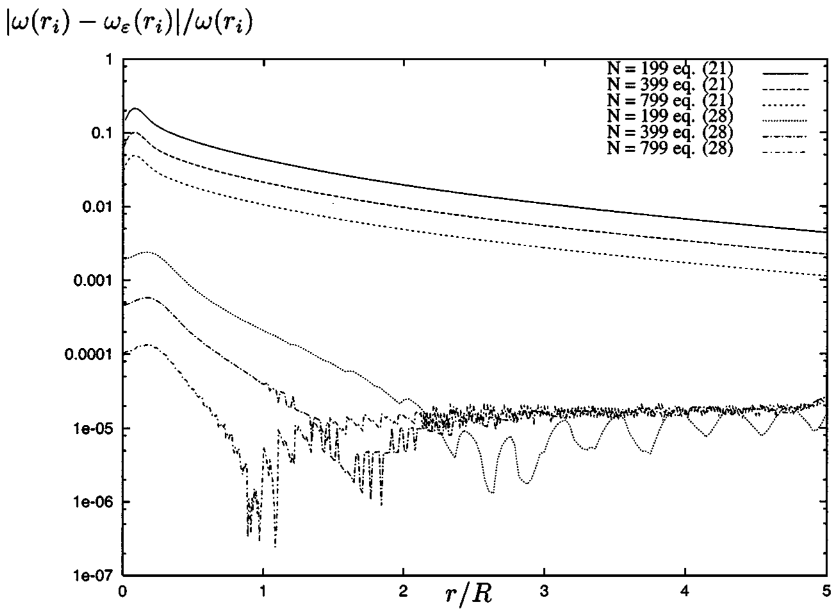

FIG. 1. Numerical results of a pure one-dimensionnal axisymmetric diffusion using the two different discrete form of the PSE method (Eqs. (21) and (28)). Evolution of the error of the vorticity along the r-axis for different discretization at $t\nu/R^2 = 0.4025$ after 40 time steps. The cutoff number ε is fixed to 0.2 and the time step $\Delta t\nu/R^2 = 0.001$.

field using the two previous formulations of the PSE for vorticity. The cutoff number and the time step are fixed, and we found that the best result is obtained by solving Eq. (28) in which the flux of vorticity on the axis is well approximated. The vorticity field is plotted on Fig. 2 to compare the analytical solution and the numerical results for different discretisations.

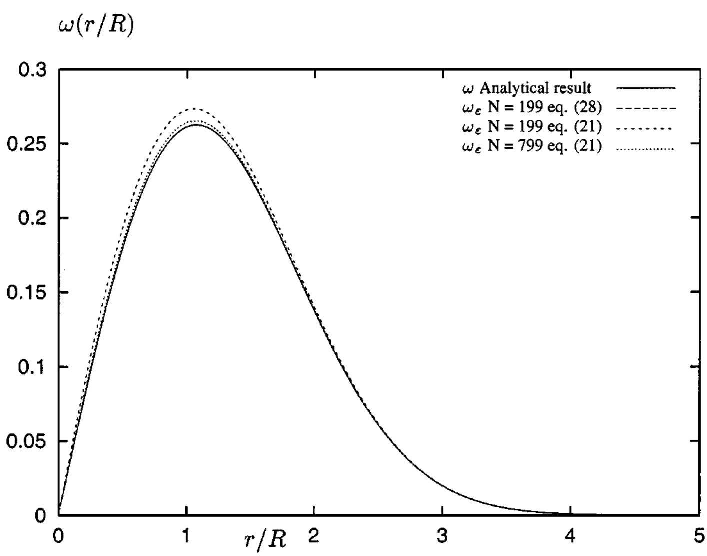

FIG. 2. Evolution of the vorticity ω versus r/R. Comparaison between the two different discrete form of the PSE method (Eqs. (21) and (28)) and the analytical result at $t\nu/R^2 = 0.4025$ after 40 time step. The other parameters are the same as in Fig. 1.

6. CONCLUSION

The numerical solution of a diffusion equation by means of particle method has been performed according to two different discretisation schemes which are theoretically equivalent. The two models differ essentially in the discretisation of the vorticity flux across the axis. It has been observed that the method in which this flux is explicitely computed shows better conservation properties.

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